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Holography and T-duality

Marco Laucelli Meana and Jesús Puente Peñalba

Dpto. de Física, Universidad de Oviedo
Avda. Calvo Sotelo 18
E-33007 Oviedo, Asturias, Spain
E-mail: laucelli@string1.ciencias.uniovi.es,
jesus@string1.ciencias.uniovi.es

Abstract: We use the AdS/SYM correspondence to study the relevant effects of compactified dimensions on the D-brane dynamics. We present a detailed picture of the T-duality transition between branes in type-IIA and type-IIB supergravity. An analysis of the renormalization scheme coming from the expectation values of background fields and the role of Wilson lines in it is given. We finally explore finite size effects and T-duality maps on the description of Wilson loops by supergravity.

Keywords: D-branes, String Duality, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

The celebrated AdS/CFT conjecture [1, 2, 3] opened, two years ago, an important path to study strongly coupled gauge theories from an unexpected supergravity framework. With this new technology, the string theorists have been able to describe a variety of properties of gauge theories unreachable from the field theory point of view (see [4]). All this work was supported by the conformal invariance of the $\mathcal{N} = 4$ SYM on the D3-brane and represented in supergravity by the conformal nature of the $AdS_5 \times S^5$ of the corresponding background. This symmetry is a strong constraint on the systems that has allowed a good description of it.

The question we will approach in this paper is in the direction of the extension of the Maldacena's conjecture to non-conformal branes. Some work on the subject has been done [5]. In particular we will focus on the compatibility of a string symmetry, T-duality, with the holographic conjecture. There have been some approaches to the problem [6, 7] and it seems that, at least, part of the phase space of branes on tori has been constructed [8, 9].

Our first interest in this paper is the clarification of such results. We will argue that T-duality does not affect the holographic description of D-brane dynamics, and we could map from one brane system to the other, any physical property of the system. We will specially focus on the renormalization of classical parameters, and discuss how it must be understood.

We will explore the corrections of the dynamics due to the presence of compact dimensions. This analysis will allow us to present of a detailed description of different parts of the phase space. In particular we describe an intermediate phase between two T-dual systems that is exclusively produced by finite size effects.

Finally we will explore the dynamics of Wilson loops in this kind of backgrounds. The analysis of this system will be done with the same philosophy of the previous sections. We will study two different quark-antiquark configurations and see how T-duality and finite size effects change their properties.

2. The background solution

We are interested in obtaining the complete background solution for wrapped D-branes and their T-duals, that is, unwrapped D-branes moving in a transverse space that is compactified in at least one direction.

We shall compute one (the latter) and deduce the other with the help of the Buscher rules for T-duality[10]. For definiteness, let us concentrate in the case of D2-branes with a transverse space like $\mathbb{R}^6 \times S^1$. It is easier if we adopt the multi-centered image, which consists in considering the circle as the whole $\mathbb{R}$ space with an identification $x \sim x + 2\pi R$. This means that the compactified metric is equivalent to the one induced by an infinite number of parallel and equally-spaced D2-branes. Of course, the solutions cannot be linearly added because Supergravity is not at all a linear theory. However, happily enough, we can write everything in terms of a function, usually called harmonic function, that does behave linearly [11]. All physical fields, in particular, the metric and the dilaton, are non-linear functionals of it.

The general p-brane solution is

$$ds^2 = f^{-1/2}(-dt^2 + dx_i^2) + f^{1/2} dy_j^2 \,, \tag{2.1}$$

with

$$e^{\phi} = g_s f^{\frac{3-p}{4}} \,, \tag{2.2}$$

where i is the index of the directions parallel to the brane, f is the harmonic function that depends on the transverse coordinates and on the radius of the compact direction. This function f obeys a Laplace equation whose solution, for a group of N superposed D2-branes is [11]

$$f(r) = 1 + \frac{g_s N l_s^5 d_2}{r^5} \,, \tag{2.3}$$

where $d_2 = (2\pi)^5 \Gamma(7/2)/10\pi^{7/2}$. The multi-centered solution is

$$f(r) = 1 + \sum_{n=-\infty}^{\infty} \frac{g_s N l_s^5 d_2}{\left[r_\perp^2 + \left(r_\parallel + 2\pi n R\right)^2\right]^{5/2}} \,. \tag{2.4}$$

The subscripts separate the coordinate that parameterizes the circle from the other transverse ones. Now we take the decoupling limit [1]to see if we get something similar to an AdS space. We must define $u_\perp := r_\perp l_s^{-2}$, $u_{||} := r_{||} l_s^{-2}$, and also $\mathcal{R} := R l_s^{-2}$. We take l_s and all other lengths to zero but keeping the u and $\mathcal{R}$ variables finite. The first two follow the usual infrared limit that is necessary to decouple the open strings attached to the branes from the closed strings. In the supergravity, this separates the fields moving in the minkowskian geometry very far from the horizon from the fields very near it. The meaning of the definition of $\mathcal{R}$ is not the same; r and u are variables and the limit is a restriction of the values, on the other hand, the radius is a constant and what we are doing is to choose the value of R in such a way that $\mathcal{R}$ is finite (and much smaller than l_s^{-1}). We will later explain the reason for this choice.

The only relevant magnitudes in the new geometry will be the ones defined above. We obtain

$$f(u_\perp, u_{||}, \mathcal{R}) = \frac{g_s N d_2}{l_s^5} \sum_{n=-\infty}^{\infty} \left[u_\perp^2 + \left(u_{||} + 2\pi n \mathcal{R} \right)^2 \right]^{-5/2} . \tag{2.5}$$

It is useful to use spherical coordinates in the six-dimensional transverse space and angular ones for the compact circle. This way $\mathcal{R}$ can explicitly appear in the metric. That way the solution becomes

$$ds^2 = f^{-1/2}(-dt^2 + dx_i^2) + f^{1/2}\mathcal{R}^2 d\theta^2 + f^{1/2} du_\perp^2 + u_\perp^2 f^{1/2} d\Omega_5^2 \,. \tag{2.6}$$

$\theta = u_{||}/\mathcal{R}$ is dimensionless and has periodicity 2π. The three longitudinal coordinates and $u_\perp$ form a kind of AdS space over which a sphere and a circle, both with variable radii, are fibered. The T-dual metric is obtained with the substitution

$$f^{1/2}\mathcal{R}^2 d\theta^2 \longrightarrow f^{-1/2}\mathcal{R}^{-2}\alpha'^{-2} d\psi^2 \,. \tag{2.7}$$

The way we have dealt with the solution does not seem to be symmetric with respect to the duality. We have constructed the solution deforming the uncompactified solution of the D2-branes, but in the limit $\mathcal{R} \to 0$ we should recover the D3-brane case. There should, then, exist an expression that deforms the D3-brane solution adding the contribution of the closed strings wound around the compact Dirichlet direction, instead of the contribution of the mirror images, as we have done. It can be obtained by a Poisson resummation of the function f.

$$\begin{aligned} f(u_\perp, \theta, \mathcal{R}) &= \frac{g_s N d_2}{l_s^5} \sum_{n=-\infty}^{\infty} \left[u_\perp^2 + \mathcal{R}^2 \left(\theta + 2\pi n \right)^2 \right]^{-5/2} \\ &= \frac{g_s N d_3}{l_s^5 \mathcal{R}} u_\perp^{-4} + \sum_{n=1}^{\infty} \frac{g_s N d_3}{l_s^5 \mathcal{R}^3} u_\perp^{-2} \cos(n\theta)\, K_2\left(\left| \frac{n u_\perp}{\mathcal{R}} \right| \right) , \end{aligned} \tag{2.8}$$

where $d_3 = 4\pi$. The first term of the sum is the solution for the D3-brane with coupling $g_s' = g_s/(l_s\mathcal{R})$. Notice that the natural transition point is $u_\perp \sim \mathcal{R}$. By 'transition point' we mean the value of the variables where the zero mode (first term) of one of the series begins to be a bad approximation as more and more terms need to be added to get a certain accuracy, and the zero mode of the other series begins to be a good approximation.

The physical meaning of this resummation is something similar to the Gregory-Laflamme [8, 12] localization transition of black holes. Here we do not have a horizon, and therefore, neither have we a definite size. However, at a certain transverse distance (in $u_\perp$) of the center, the D2-brane solution has a width in the longitudinal direction ($u_\|$) that can define a typical size, precisely of order $\mathcal{R}$. When the distance to the center is larger than that, the metric is basically constant with respect to $u_\|$. In the picture where the circle is emulated by a one-dimensional lattice, the observer beyond the point $u_\perp = \mathcal{R}$ sees a nearly continuous distribution of branes in the compact direction. Any computation obtained as a sum of contributions of each individual D2-brane has to be resummed.

We expect to relate this soft transition to T-duality so it is somewhat surprising that α' does not appear in the condition. The explanation is as follows. There are two different reasons to prefer a theory instead of its T-dual. In the case of the closed strings, there are two series of energy levels in the spectrum, windings and momenta. If the winding modes are much lighter than the momenta, it is necessary to resum the perturbative series and therefore, it is more useful to use the T-dual theory. When $R = \sqrt{\alpha'}$, both modes have the same energy and that is why that is the limit of usefulness of each theory. On the other hand, when only open strings are involved, windings and momentum cannot coexist. Only when there is another length (or energy) scale like the separation of branes is there a reason to prefer one of the two theories. If the gaps in the discrete spectrum are much smaller than the separation of the branes many of them will contribute to a typical process and it is better to use the theory where they are momenta and can be easily integrated. Otherwise, one should use the other one. The range where the transition happens is around $R = Y$, if Y is the separation between the branes.

This second possibility is the one we have found in this section.

3. The gauge theories

Let us now see which is the field theory that is described by this geometry. It is the one that appears when we take the same limit in the open string model of the same group of D-branes. The modes of the string that do not decouple are the massless ones, whose dynamics is governed by the SYM_{2+1}; and the strings that wind a finite number of times around the compact direction before both ends attach on the D2-brane. Again, we should use the multi-centered image, which is the clearest. The field theory is a $\mathrm{U}(N \times \infty)$ SYM in $2+1$ dimensions. N is the number of parallel and superposed D2-branes, it must be large in order for the conjecture to work well. The infinity that multiplies that N can be substituted by any number large enough to describe all the windings of the system. The scalar with index in the compact direction takes the following expectation value

$$X^i = \mathrm{blockdiag}\{\ldots, -Id_{N\times N}, 0_{N\times N}, Id_{N\times N}, \ldots\} \tag{3.1}$$

that breaks the gauge symmetry to the subgroup $\mathrm{U}(N)^\infty$.

There is a T-dual description of this system that interchanges windings and momenta, which are, in this case Higgs' masses and Kaluza-Klein momenta. It is the SYM_{3+1} compactified in a circle of radius $\tilde{R}$ (that equals $\alpha'/R = \mathcal{R}^{-1}$). The isomorphism identifies the fields of the $N \times N$ boxes which are, say, M places away from the diagonal with the $N \times N$ fields that complete the adjoint representation of the SYM_{3+1} with KK momentum $M/\tilde{R}$ ($=M\mathcal{R}$). There are two ways to write the action of this theory: as a three-dimensional theory with a compact direction or as a two-dimensional theory with an infinite number of fields of relatively integer masses. When the radius $\tilde{R}$ is small, or better, when the typical energy of the experiment is smaller than the first KK level, the two-dimensional representation is more suitable while in the opposite case, one should use the three-dimensional action.

Both actions reproduce the same physics and, therefore, give the same results. Renormalized parameters are functions of f, for example

$$g_{YM} = e^{\phi/2} = g_s^{1/2} f^{1/8}\,. \tag{3.2}$$

The interpretation of the sum we have written in formula (2.8) is that it is the way the masses originated by the compactification add different corrections to the renormalization of the two uncompactified theories and interpolate between them in a continuous manner. There is, however, one subtlety. The interpolation is soft between the two behaviours of the function f, but in the solution of both the dilaton (the gauge coupling) and the metric there is a discrete jump: in the first case, there is a different power of f and in the second there is an inversion of one of the components. The meaning of this is that the two different actions that describe the field theory do it in terms of two kinds of fields. The two- and three-dimensional fields have different dimensions and therefore do not scale equally.[1] In particular, the gauge coupling of the three-dimensional theory is dimensionless and, classically, it is independent of the scale; on the other hand, in the theory with two dimensions, the coupling is a

[1] As usual [3] computing two-point functions of reduced fields on the transverse sphere one finds that the weights of three-dimensional fields correspond to masses on the 3-brane background. For two-dimensional fields we should reduce fields of type-IIA supergravity on $S^1 \times S^5$, and their masses should account for their weight. In this case both the supergravity and gauge theory fields hold in representations of $\mathrm{SU}(4) \times \mathrm{U}(1)$ R-symmetry group.

length to the 1/2 power and the dimensionless coupling $(g_{YM}\sqrt{U})$ depends on the scale. In principle, both descriptions are not only equivalent, but, in fact, are exactly the same. In both cases loops take the form of sums in a discrete number which we can call momentum or winding, but it is the same thing. So the picture is that the effective coupling when, for example, the energy is very low, is that of the D2-brane theory. If we are interested in higher energies, we reach a region in phase space where there is an increasing number of addends in the function f that contribute significantly, and when they are neatly higher than $\mathcal{R}$, the series can be resummed to obtain a new good zero mode. In terms of fields, this resummation can be seen as the redefinition of two-dimensional fields in terms of three-dimensional ones:

$$A^i(x_3, x_j) = \sum_{n=-\infty}^{\infty} X_n(x_j) e^{-inx_3/\tilde{R}}\,. \tag{3.3}$$

This change of variables is not trivial at all when one considers the renormalized theories because R, as any other parameter appearing in the lagrangean acquires a dependence on the scale u, that we should introduce in the Fourier expansion. This is a consequence of the renormalization of the masses of the KK modes. The redefinition affects the coupling constant as it does in any dimensional reduction:

$$(g_{YM})^2 = \frac{(g'_{YM})^2}{R_{\rm ren}(u)}\,. \tag{3.4}$$

This means that the behaviour under renormalization of both theories can be very different. In our case, using the correspondence, one finds

$$u_\perp^{-5/8} \overset{u\ll\mathcal{R}}{\longleftarrow} g_{YM} \overset{u\gg\mathcal{R}}{\longrightarrow} u_\perp^{-1/2}\,, \tag{3.5}$$

while g'_{YM} is independent of the scale. This is due to the fact that the dependence on the scale of g_{YM} and that of $R_{\rm ren}(u)$ exactly cancel. In fact, the behaviour $g_{YM} \propto u_\perp^{-1/2}$ is conformal because the dimensionless coupling in two dimensions $g_{YM}\sqrt{u_\perp}$ is, indeed, independent of the scale.

It is interesting to notice that in this case the map that relates both sides of the correspondence is a little more complex. Indeed, the renormalized magnitudes depend on three parameters ($u_\perp$, θ and $\mathcal{R}$) instead of one. From the original correspondence for the D2-brane case we know that $u_\perp$ is the μ parameter, the typical energy of the experiment which we are testing the system with. The reason why $\mathcal{R}$ appears in the D2-brane side is that the renormalization that the supergravity gives uses a mass-dependent scheme. The massive fields give contributions to the renormalized quantities in such a way that when energies (μ) are smaller than their mass, they naturally decouple due to the fact that the terms of the series related to them tend to zero in that limit. In our case, those terms are lorentzian-like. In a mass-independent scheme, the terms are added by hand when necessary. The translation would be

$$\left[u_\perp^2 + \mathcal{R}^2\left(\theta + 2\pi n\right)^2\right]^{-5/2} \longrightarrow u_\perp^{-5}\mathcal{H}\left(u_\perp - \mathcal{R}\left|\theta + 2\pi n\right|\right), \tag{3.6}$$

where $\mathcal{H}$ is a Heaviside step function. Although the function tends to a constant, not to zero, when the variable is small, it is possible to neglect it because it is much smaller than the divergent zero mode term. The variable θ gives a contribution to the mass because from the point of view of the field theory, it is the expectation value of the scalar field that is used as a test; this breaks the gauge symmetry and changes the mass of some fields.

In the picture with the D3-branes, there are not any masses but discrete momenta. We know that in mass-independent schemes, compactification does not affect the renormalization and no dependence on the radius appears. The reason is that mass-independent renormalization adds the minimum counterterms to the Feynman diagrams to cancel the ultraviolet divergences. Obviously, compactifying does not affect at all the ultraviolet behaviour because it is a purely infrared phenomenon. However, if one wants to observe how the infrared effects act over the effective coupling, one should use a radius-dependent scheme and that is exactly what the supergravity result is. Indeed, the first term is the one given by the minimum counterterms and the rest are exponentially suppressed when the energy is larger than the inverse radius. In this case θ is related to the Wilson line. The precise identification is

$$A^i = \frac{\mathcal{R}}{2\pi}\theta\,. \tag{3.7}$$

The reason why it appears as a renormalization parameter is that the interaction, and therefore the coupling constant is affected by its presence. Its effects are periodic, that is why they appear as a Fourier expansion, and depend on the proportion between $u_\perp$ and $\mathcal{R}$. If the radius is very small, then the exponential asymptotic behaviour of the Bessel functions makes the contribution of the Wilson line negligible, but if the radius is large, then the function f develops a clear maximum in $\theta = 0$ and also does the coupling. This is due to the fact that the Wilson line breaks the gauge symmetry and gives masses to the intermediate (off-diagonal) bosons. In the infrared, when $u_\perp$ is much smaller than $\mathcal{R}$, these masses can forbid any interaction and that is why the effective coupling can tend to zero and is maximum when the Wilson line is not present at all ($\theta = 0$).

4. Effects of renormalization on the phase space

As we have written just above, not only the coupling changes with the 'distance' u, also the radii of both the five-sphere and the circle change and it is interesting to analyze its meaning. Results related to this section are in [5, 8].

Let us start with the circle.[2] In the supergravity, $\mathcal{R}$ is a parameter related to the value of the radius measured by a faraway observer situated where the metric is

[2]We are very grateful to María Suárez for her help on the topics covered in this section.

minkowskian. The limit we have taken has completely disconnected that observer from the system we want to study and the physical radius is a function given by

$$R_{\text{phys}}(u_\perp, \theta, \mathcal{R}) = \alpha' \mathcal{R} f^{1/4}(u_\perp, \theta, \mathcal{R}) \tag{4.1}$$

for the D2-brane case and the inverse for the D3-brane one.

The interpretation of this is direct. In any field theory all parameters, including radii can be renormalized. The expression above tells us how the physical, renormalized radius depends on the scale. This can also be seen as the renormalization of the masses of the Kaluza-Klein modes.

Let us now discuss the low energy spectrum of the two (IIA and IIB) T-dual string background solutions that we have. We are interested in particular in the energies of winding modes and discrete momenta. Classically, if we choose $\mathcal{R} = R/l_s^2$ to be an order one magnitude, then the energy of the IIA windings is also order one ($2\pi m\mathcal{R}$), while that of the momenta is large ($\mathcal{R}^{-1} l_s^{-2}$) and decouples. If we now look at the physical radius, because of the fact that the function f runs from 0 to infinity, this is not true in all points of space. In fact, there is a point where $R_{\text{phys}}(u_\perp, \theta, \mathcal{R}) = l_s$ where momenta begin to be the lightest modes and must be taken into account while windings decouple.

This is the usual phenomenon that sets the limit of utility of two T-dual closed string theories, as we have written before. In this case, this soft transition always happens when $u \gg \mathcal{R}$, that is, in the far ultraviolet of the Yang-Mills theories. Explicitly

$$R_{\text{phys}}(u, \theta, \mathcal{R}) = l_s \qquad \text{if} \qquad u_\perp \simeq d_3^{1/4} (g_s N)^{1/4} l_s^{-1/4} \mathcal{R}^{3/4}\,. \tag{4.2}$$

That limiting value for u is much larger than $\mathcal{R}$ because $(g_s N)/l_s$ tends to infinity in order for the conjecture to work. For this reason, this transition, always occurs when f can be well approximated by

$$f(u_\perp, \theta, \mathcal{R}) = \frac{d_3 g_s N}{l_s^5 \mathcal{R} u_\perp^4} \tag{4.3}$$

that is the harmonic function of a D3-brane in an open space. When $u_\perp$ is beyond both transition points, in the ultraviolet, the D3-brane solution is exact and finite size effects are negligible, they become smaller and smaller as $u_\perp$ grows and the theory approaches an ultraviolet fixed point. When it is smaller ($u \ll \mathcal{R}$), it is the D2-brane solution the one that describes well the problem. When

$$\mathcal{R} < u < (g_s N)^{1/4} l_s^{-1/4} \mathcal{R}^{3/4} \tag{4.4}$$

there is an intermediate case. One can use the supergravity solution of the D3-brane, but size of the circle is such that the momenta are light enough to be considered continuous but windings (which are string corrections) are even lighter and have to be added. On the other hand, one could use T-duality and find the solution in terms of D2-branes; then windings are heavier than momenta, as they should, but still, the harmonic function must be resummed. In the picture where the circle is seen as $\mathbb{R}/\mathbb{Z}$, the solution is that of a continuous set of D2-branes that fill one of the perpendicular directions.

The question now is how can we interpret this transition from the point of view of the Yang-Mills theories. The guides are group theory and the identification of $R_{\text{phys}}(u)$ as the renormalized value of $\mathcal{R}$. The supergravity fields that have discrete momentum in the IIB theory (or winding in the IIA one) clearly correspond to the Kaluza-Klein modes of the SYM_{3+1} (or the massive off-diagonal modes in the $\text{U}(N \times \infty)$ SYM_{2+1}). The relation is the charge under the U(1) symmetry of the circle.

Compactifying a Yang-Mills theory in a circle has several consequences: the momenta get discrete and the component of the gauge field polarized in that direction acquires a periodicity $A^i \sim A^i + 1/R$, besides that field can have a vacuum expectation value (Wilson line). The periodicity affects the momentum of the field (the electric field in that direction) making it discrete with gap $2\pi R$. The electric field has energy proportional to $\vec{E}^2$. The renormalization of $\mathcal{R}$ makes this energy spectrum identical to that of the windings around the compact direction (in the IIB 3-brane supergravity). We can be more precise. Let us design an experiment with an observer in the IIA supergravity theory moving in the direction of the circle, that is one D2-brane, separated from the rest, falling towards the central bundle of N D2-branes. From the point of view of the $\text{U}(N+1)$ SYM_{2+1}, the movement is described by a time-dependent vacuum expectation value of the scalar that breaks the $\text{U}(N+1)$ into $\text{U}(N) \times \text{U}(1)$. This represents the change of the coordinate of the D2-brane as it moves. With the usual identification, that velocity is related in the $\text{U}(N+1)$ SYM_{3+1} to the value of a background electric field. As we have taken our observer to be a D2-brane, its movement is Galilean and that is why the dispersion relation energy-electric field is quadratic.

With this we learn a bit about the meaning of the closed string T-duality for the correspondence. If in usual string theory it could be seen as a competence between momenta and windings, here it can also be seen as a competence between two different spectra. In perturbative gauge theories there are two kinds of excitations: the perturbative fields and the solitons that represent changes in the vacuum. In our case the vacuum is parameterized by the expectation values of the scalars. Usually, unable to do exact computations, one uses the Born-Oppenheimer approximation and consider the solitons as 'slow modes' and the perturbative fields as 'fast modes'. Their dynamics have so different frequencies or time scales that completely decouple and one can solve them independently. In our case, the D-branes are so heavy that their movements and those of the open strings are independent. In our case we have found that when a Yang-Mills theory (at least of the kind we are studying) is compactified, depending on the value of the energy in terms of the radius and the other parameters (g_s and N) both types of modes (polarized in the compact direction) can interchange the role of being the lightest. When the energy is high, the slow modes are very heavy while the fast ones are light, but below the transition point written in (4.2), the situation is the opposite.

To describe the intermediate phase we can either use the U(N) SYM$_{3+1}$ with one compact direction, light KK modes but even lighter electric fields or (better we would say) the U($N\times\infty$) SYM$_{2+1}$ with the gauge symmetry slightly broken by some light off-diagonal modes.

Once we have dealt with the radius, let us see now what happens to the couplings. They are

$$g_{\rm IIB}(u_\perp, u_\|, \mathcal{R}) = g_s^{\rm IIB} \qquad \text{and} \qquad g_{\rm IIA}(u_\perp, u_\|, \mathcal{R}) = f^{1/4} g_s^{\rm IIA} = f^{1/4} g_s^{\rm IIB} \mathcal{R} l_s\,. \quad (4.5)$$

As the IIB string theory is S-selfdual, we can choose $g_s^{\rm IIB}$ to be smaller than one. However, we cannot prevent that $g_{\rm IIA}$ become larger than one in some region in the target space, when

$$u < \left(g_s^{\rm IIA}\right)^{5/4} N^{1/4} \mathcal{R}^{-1/4} l_s^{-5/4}\,. \quad (4.6)$$

There, it is necessary to consider corrections from M theory.

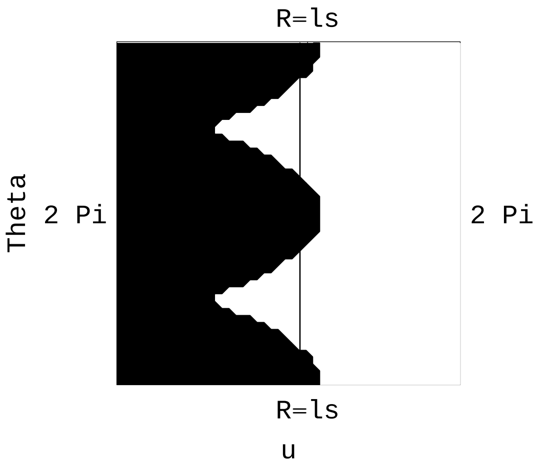

Figure 1: T-dual phases in terms of the value of the physical radius. We see how finite size effect changes the form of the transition line $R_{\rm phys} = l_s$. The straight line represents the transition without considering finite size effects.

Another limit is imposed by the variable radius of the sphere. When it is too small, the curvature can exceed the string scale and then it is also necessary to add corrections. This is not different from the usual uncompactified cases so we shall not discuss it more. In the picture 1, we show the effect of the Wilson line (θ parameter) on the phase space. The line that separates the two phases, the one describable in terms of D3-branes and the other in terms of the continuous set of D2-branes, is not a straight line. The Wilson line extends or reduces each phase depending on its value. More to the left, we could draw a second wavy line, parallel to this one, that signalled the appearance of an M2-brane phase when the coupling constant is larger than one. The phase with D2-branes in an open space ($u_\perp < \mathcal{R}$) can exist or not. It does exist if the M2-brane phase appears precisely when $u_\perp < \mathcal{R}$. The line of phases is, therefore: wrapped D3-brane, continuous set of D2-branes, D2-brane in open space (not always) and M2-brane. The straight line drawn in figure 1 is the result obtained with the D3-brane neglecting all finite volume effects.

5. Wilson loops

In this section we will analyze Wilson loops in theories with a compactified dimension. As in the rest of our work we will deal with the D2 and D3-brane systems. We will focus our attention on those features coming from the presence of a compactified dimension.

Let us remind how we can introduce static quarks in gauge theories living on the brane. We begin with $N+1$ branes and we break the $\mathrm{U}(N+1)$ gauge theory by taking one brane to infinity. The open strings connecting the separated brane to the others have infinite mass and transform in the fundamental (antifundamental) of the gauge group. These are the external quarks. From the worldvolume point of view these sources produce solitonic deformations on the worldvolume because the N-branes are pulled by the infinitely long open strings attached to the other.

The shape of this deformation could be represented, on the supergravity side, by the world-sheet of a string ending on the boundary of the background manifold. Using this prescription we can stablish a concrete correspondence between the quark-antiquark potential and the classical action of the string as [13, 14]

$$\langle W(C)\rangle_{\text{Gauge}} = A(L)e^{-TE(L)} = \langle e^{-S}\rangle_{\text{Sugra}} \tag{5.1}$$

where C is the Wilson contour and L is the distance from one source to the other. The string action is computed as

$$S = \frac{1}{2\pi\alpha'}\int dx dt\sqrt{\det G_{mn}\partial_a X^m \partial_b X^n}\,, \tag{5.2}$$

where G_{mn} is the euclidean background metric in the loop directions. In order to compute the classical action we should find the string configuration that minimizes the world-sheet area.

We are interested in finite size effects due to the compact dimension. We will investigate how to compute the Wilson loop in the D2-brane case and compare it with the result obtained in the D3-brane theory. There are several situations to be considered and we will describe the T-duality map for each case. These situations are related to different loop geometries.

Suppose a quark-antiquark pair in the D2-brane theory separated in a world-volume direction. Remember that in this case the boundary theory has a $\mathrm{SO}(6)\times\mathrm{U}(1)$ R-symmetry that corresponds to rotational symmetry on the supergravity side. This allows us to take quarks on different points on the $S^5\times S^1$ transverse manifold and introduce an angular difference between the open strings connecting the separated branes and the N central ones.

We will begin with the simplest configuration that corresponds to taking the angular difference to zero. Using the metrics in (2.6) we can see that the classical world-sheet action is given by

$$S=\frac{T}{2\pi\alpha'}\int dx\sqrt{G_{tt}G_{xx}+G_{tt}G_{uu}(\partial_x u)^2}\,, \tag{5.3}$$

where we have integrated the temporal variable assuming a static configuration for the loop.

We can compute the worldsheet area in (5.3) using a general result as, for example, in [16]. These general formulas allow us to stablish that if the Nambu-Goto action is written as

$$S=\frac{T}{2\pi\alpha'}\int dx\sqrt{A(u)(\partial_x u)^2+\frac{B(u)}{\lambda^4}} \tag{5.4}$$

then the physical magnitudes that we are interested in could be written

$$\begin{aligned} L &= 2\lambda^2\sqrt{B(u_0)}\int_{u_0}^{\infty}du\sqrt{\frac{A(u)}{B(u)(B(u)-B(u_0))}}\,,\\ E &= \frac{1}{\alpha'\pi}\int_{u_0}^{\infty}du\left[\sqrt{\frac{A(u)B(u)}{B(u)-B(u_0)}}\right]-\frac{1}{\alpha'\pi}\int_{u_{min}}^{u_0}du\sqrt{A(u)}\,, \end{aligned} \tag{5.5}$$

where u_0 is the point given by $u(x=0)=u_0$, and u_{min} is a geometrical minimal value of u given, for example by the presence of a singularity on the background manifold.

We can now compute the Wilson loop in the D2-brane theory. Using the action in (5.3) and expressing it in terms of the functions in (5.5) we can write

$$\begin{aligned} A(u_\perp) &= G_{uu}G_{tt}=\alpha'^2\,,\\ B(u_\perp,u_\parallel,\mathcal{R}) &= \lambda^4 G_{xx}G_{tt}=\alpha'^2 H(u_\perp,u_\parallel,\mathcal{R})^{-1}\,, \end{aligned} \tag{5.6}$$

where $\lambda^4=d_2 g_{YM}N$ and $H(u_\perp,u_\parallel,\mathcal{R})$ is the sum over multicenter solutions in the first line of (2.8). Finally one obtains

$$\begin{aligned} L &= 2\lambda^2\int_{u_0}^{\infty}du\frac{H(u,u_\parallel,\mathcal{R})}{\sqrt{H(u_0,u_\parallel,\mathcal{R})-H(u,u_\parallel,\mathcal{R})}}\\ E_{qq} &= \frac{1}{\pi}\int_{u_0}^{\infty}du\sqrt{\frac{H(u_0,u_\parallel,\mathcal{R})}{H(u_0,u_\parallel,\mathcal{R})-H(u,u_\parallel,\mathcal{R})}} \end{aligned} \tag{5.7}$$

The final scope of our computation should be the resolution of E in terms of L. It does not seem possible to obtain this result analytically but it is possible to do it numerically. We can compute both integrals in terms of the variable u_0 and then extract a numerical representation of $E=E(L)$. We will recover the analytic result of the Wilson loop in the limit $\mathcal{R}\to\infty$, which corresponds to the D2-brane on a decompactified background.

The complete T-dual map for this quark-antiquark computation is obtained by computing the Wilson loop from the D3-brane point of view. The method is exactly the same. In this case we will use the Poisson resummed expression for the harmonic function which is given in the second line of (2.8). We can write it as

$$B(u_\perp) = \lambda^4 f^{-1} = \alpha'^2 u_\perp^4 g(\tilde{R}, u_{||}, u_\perp)^{-1} \tag{5.8}$$

now $\lambda^4 = d_3 N g_s'$. Using this parameterization of the metric and the usual procedure we can express, as we did for the D2-brane, the energy of the Wilson loop and the separation between quarks in terms of $u_0 = u_\perp(x=0)$

$$\begin{aligned} L(u_0, \mathcal{R}, u_{||}) &= 2\lambda^2 u_0^2 \int_{u_0}^\infty \frac{du}{u^4} \frac{g(u)}{\sqrt{g(u_0) - g(u)\left(\frac{u_0}{u}\right)^4}}\,, \\ E_{qq}(u_0, \mathcal{R}, u_{||}) &= \frac{1}{\pi} \int_{u_0}^\infty \frac{du}{\sqrt{g(u_0) - g(u)\left(\frac{u_0}{u}\right)^4}} - \frac{1}{\pi}\int_{u_0}^\infty du\,, \end{aligned} \tag{5.9}$$

where the last integral eliminates the usual divergence coming from the quark masses. We can recover the standard D3-brane result simply taking the radius parameter $\tilde{R}$ to infinity then reducing the complete series in $g(\tilde{R}, u_{||}, u_\perp)$ to its zero mode.

The first evidence that comes out directly from the results in (5.7) and (5.9) is that both solutions are exactly the same, but expressed in T-dual variables. This fact simply reflects that, for this simple configuration of the Wilson loop, T-duality does not affect any parameter defining the system. More concretely, what we have seen is that the effective metric, used to compute the string worldsheet area, remains unchanged under Busher's transformation rules. Consequently, they give the same result for the quark-antiquark potentials in D2 and D3 brane theories.

In the figure 2 we have plotted our result. We have normalized the variables in the integrals, working with dimensionless parameters. The election has been such that the results can be written in terms of $\mathcal{E} = E_{qq}\tilde{R}$ as a function of $\mathcal{L} = L\tilde{R}^{-1}$.

Assuming the usual functional form

$$\mathcal{E} \sim \mathcal{L}^{a(\mathcal{L})} \tag{5.10}$$

we can use the variable $a(\mathcal{L})$ to see how the quark-antiquark potential goes from the unwrapped D3-brane behavior, $a(\mathcal{L}) = -1$, to the expected for the D2-brane in a decompactified background, where $a(\mathcal{L}) = -2/3$ [15]. The complete result is shown in figure 2. We see that when the distance between quarks is large enough compared with the compactification radius, $L \gg R$, the system feels a small compact dimension so the propagator of gauge fields going from one quark to the other does not include any KK mode. In this case we approach the T-dual system, that is the D2-brane. When the radius is large, or the quarks are very close to each other, $R \gg L$, the KK modes are very light and all of them must be taken into account. Then the system approaches the unwrapped D3-brane one.

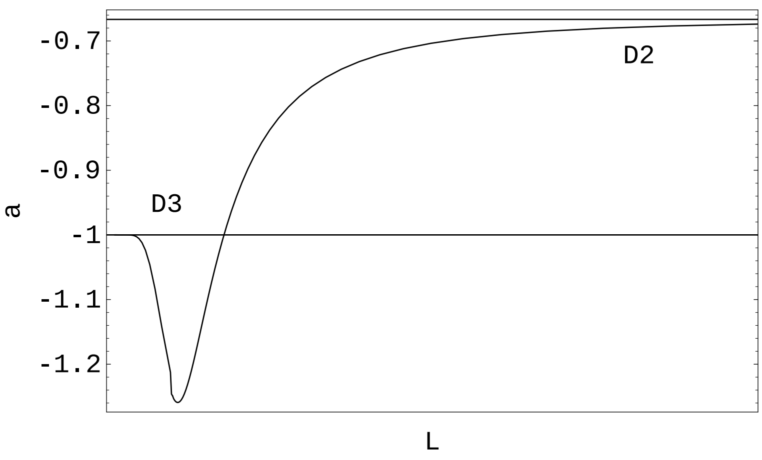

Figure 2: $a(\mathcal{L})$ in terms of $\mathcal{L}$. Here we see how the finite size effects interpolate between the D2 and D3 branes.

Let us now explore the other loop geometry we are interested in. Apart from the spatial separation between quarks, one can consider that they have a different 'flavour', that is, that, taken as vectors in the R-symmetry space $S^1 \times S^5$, the quarks point at different directions. This affects the Wilson loop and, therefore, the potential. We will concentrate on phase differences in the circle, which is the new parameter here. The angular difference on the sphere [13, 14] does not play any role in the T-duality map. The way to calculate in this case is the same as before, but adding some new terms. Again, we have to compute the area of the world-sheet, which is given by the Nambu-Goto action. It is such that

$$\begin{aligned} X^i(x,\tau) &= xL\,, \\ \theta(1,\tau) - \theta(0,\tau) &= \theta \\ T &= \tau \end{aligned} \tag{5.11}$$

and the configuration is static so nothing depends on τ. The form of the world-sheet in the target space is the same as before, except that the spatial sides of the rectangle are not exactly in the i^{th} direction but in a general direction in the (X^i, θ) plane. The action is given by

$$S = \frac{1}{2\pi\alpha'} \int dx \sqrt{G_{xx}G_{\tau\tau} + G_{uu}G_{\tau\tau}(\partial_x u)^2 + G_{\tau\tau}G_{\theta\theta}(\partial_x\theta)^2} \tag{5.12}$$

We would like to compare this expression to its T-dual. In order to do that, we have, firstly, to look for a configuration that is the T-dual of this one. It is important to notice that the dual magnitude to an angle in one circle is not another angle in the dual circle because that would mean that movements, and therefore momenta in both circles would be related by the duality, which is not true. A string stretched between two points placed a distance apart in the circle of the D2-brane solution has an energy related to its fractional winding number; so we expect that the T-dual configuration has some fractional momentum in the IIB supergravity. From the gauge theory point of view, the R-symmetry separation of quarks on the D2-brane should be represented by a pair of quarks, living on the T-dual D3-brane, moving with different momenta in the compact direction.

The main difference of the string worldsheet that describes the Wilson loop in supergravity, is that the compact scalar must have momentum so instead of the static configuration we should have

$$\begin{aligned} X^i(x,\tau) &= xL \\ \psi(x,T_{\max}) - \psi(x,0) &= \psi \\ T &= \tau \,. \end{aligned} \tag{5.13}$$

Now the action is

$$S = \frac{1}{2\pi\alpha'}\int dx d\tau \sqrt{G_{xx}G_{\tau\tau}+G_{uu}G_{\tau\tau}(\partial_x u)^2+G_{\tau\tau}G_{\psi\psi}(\partial_\tau\psi)^2+G_{uu}G_{xx}(\partial_x u)^2(\partial_\tau\psi)^2}\,. \tag{5.14}$$

Both actions are quite different. One integrates fields in one dimension and the other in two, besides, the second has a term which is quartic in the velocities. Here we will show that both actions describe the dynamics of T-dual systems. The procedure we will use assumes that, similar to what happens in the string sigma model [17], these two actions are related by a canonical transformation.

The hamiltonians of the systems are simply computed to give

$$H_{D2} = \sqrt{G_{xx}G_{\tau\tau} + \frac{G_{xx}}{G_{\theta\theta}}P_\theta^2 + \frac{G_{xx}}{G_{uu}}P_u^2} \tag{5.15}$$

and

$$H_{D3} = \sqrt{G_{xx}G_{\tau\tau} - G_{xx}G_{\psi\psi}(\partial_\psi\psi)^2 + \frac{G_{xx}}{G_{uu}}P_u^2}\,. \tag{5.16}$$

Now we simply see that the canonical transformation should be

$$P_\theta = -i\partial_\tau\psi\,, \tag{5.17}$$

with no change in the coordinates and momenta. There is a small difference between the transformation in (5.17) and those in [17]. Here we simply adopted the rule in [17] to the euclidean case. Imposing the equality of the hamiltonians H_{D2} and H_{D3} we recover the Buscher's transformation rule for the metric

$$G_{\psi\psi} = \frac{1}{G_{\theta\theta}} \tag{5.18}$$

that finally shows that the worldsheet configurations we described below are T-dual.

JHEP02(2000)016

6. Conclusions

In this work we have studied the dynamics of D-branes sitting on backgrounds with toroidally compactified dimensions. Concretely we focused our analysis on the D3 and D2-branes with one compactified dimension. All our results are straightforwardly extensible to more general situations. We worked in the framework of the Maldacena duality, trying to clarify how T-duality enters into the holographic conjecture.

Our principal interest has been the analysis of finite size effects on the dynamics of the systems. We studied their influence on the dynamics of the brane and then, by T-duality, we showed how the possible corrections appear in the dual system.

The analysis started by a detailed study of the background geometry of a D2-brane with a compactified transverse direction. In order to describe the effect of the compact circle we used the multicentered solution of IIA supergravity. We clearly explained how the near horizon limit has to be taken. The background is described by a series of harmonic functions, the eq. (2.8), expressing the presence of an infinite tower of winding modes. We found a transition point, $\mathcal{R} \sim u_\perp$, where the accuracy of any truncation of the series is ever worse. Physically this means that at this point finite size effects become very important. It is then preferable to use a Poisson resummation formula for the series. It does not mean that we are making a T-duality transformation. In fact the system now behaves as a continuous distribution of branes along the compact direction, still staying in type-IIA string theory.

We are, in some sense, forced to make use T-duality and go to the three-brane system when the physical radius, R_{phys}, is smaller than the string scale. We showed that this point is always reached at values of $u_\perp$ larger than $\mathcal{R}$, so the system has passed through the continuous distribution-phase described above. On the other hand we see that at these energies in the T-dual system finite size effects are irrelevant.

In sections 3 and 4 we present what can be learned about the gauge theory from supergravity. We know that the coordinate dependence of the expectation value of background fields corresponds to the renormalization group flow of the corresponding quantities in the gauge theory. We saw that in our cases the renormalization results coming from supergravity appear in mass-dependent renormalization scheme. It is due to the fact that, in order to obtain information from finite size effects, we should include all the infrared degrees of freedom of the theory. Finally we showed that the angular separation of a test D2-brane from the N-branes source is mapped on the D3-brane system as a Wilson line. In this case the test object sees a constant gauge field on the source. It is remarkable that all the conclusions in these sections cannot be compared to those obtained with the perturbative Yang-Mills theory. This is standard in the AdS/SYM duality where only computing magnitudes independent of the 't Hooft coupling can one compare perturbative Yang-Mills and supergravity results. Particularly, this affects the transition points in phase space. To see how this works at finite temperature see [8].

The AdS/SYM allows the description of the strong 't Hooft coupling of the gauge theories when the string coupling is small. In the case of non-conformal brane configurations this establishes some limits on the phase space describable in terms of supergravity. In our work we dealt with two non-conformal systems. In the case of the D2-brane the renormalization is shown by the running of the string coupling and of the size of the physical radius of the compact dimension. The latter reflects the change of the masses of winding modes. The D3-brane case is more subtle. In order to see the running of the coupling one cannot use the ten-dimensional string coupling but one must use the coupling of the T-dual D2-brane. The explanation of this requirement could be seen from pure gauge theory arguments. When we compactify the one of the three-brane coordinates and express our fields in terms of KK modes in two dimensions we really deal with the D2-brane which coupling constant is the T-dual of the initial one. Another way to see the non-conformal nature of the wrapped three-brane, using fields in three dimensions, is looking at the energy dependence of their masses.

Finally we studied the dynamics of Wilson loops in the systems. We presented two possible configurations of the loop. They correspond to two different geometries of the string worldsheet that describe the quark-antiquark interaction from supergravity.

The simplest one described a pair of quarks at a distance L in the compact direction and with the same position in the angular directions. This configuration corresponds to a static string worldsheet. We studied the evolution of the properties of the system in terms of the supergravity parameters. We saw how the system goes from a pure D2-brane loop to the three-brane one. Our computation allowed us to study how the presence of a compact direction could affect the system.

The other quark-antiquark configuration we considered, includes an angular distance, on the circle, between infinite strings describing static quarks of the two-brane. In this case we showed that T-duality converts this static system into a time-dependent one on the three-brane. Concretely it corresponds to quarks moving on the compact worldvolume direction with different momenta. We constructed this configuration from physical arguments and finally showed that it is exactly the same as the initial one but expressed in canonical transformed variables.

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JHEP02(2000)016

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JHEP02(2000)016

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